Module 2: Analysis of Stress

2.3.1 GENERAL STATE OF STRESS IN THREE-DIMENSION IN CYLINDRICAL CO-ORDINATE SYSTEM

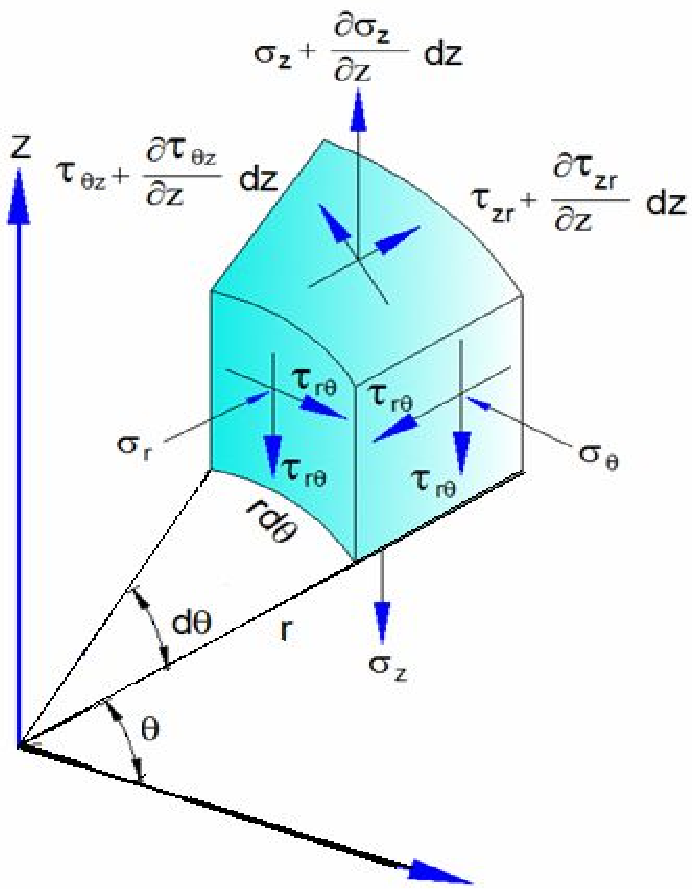


Figure 2. 17 Stresses acting on the element

In the absence of body forces, the equilibrium equations for three-dimensional state are given by

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial z} + \left(\frac{\sigma_r - \sigma_\theta}{r}\right) = 0 \qquad (2.47)$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} = 0 \qquad (2.48)$$

$$\frac{\partial \tau_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{zr}}{r} = 0 \qquad (2.49)$$

2.25 NUMERICAL EXAMPLES

Example 2.1

When the stress tensor at a point with reference to axes *(x, y, z)* is given by the array,

$$\begin{bmatrix} 4 & 1 & 2 \\ 1 & 6 & 0 \\ 2 & 0 & 8 \end{bmatrix} \; \textit{MPa}$$

show that the stress invariants remain unchanged by transformation of the axes by 45^0 about the *z*-axis,

Solution: The stress invariants are

I_1 = 4 + 6 + 8 = 18 MPa

I_2 = 4×6+6×8+4×8-1×1-2×2-0 = 99 MPa

I_3 = 4×48-1×8+2×(-12) = 160 MPa

The direction cosines for the transformation are given by

	x	y	z
x'	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0
y'	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0
z'	0	0	1

Using Equations (2.21a), (2.21b), (2.21c), (2.21d), (2.21e), (2.21f), we get

$$\sigma_{x'} = 4\times\frac{1}{2} + 6\times\frac{1}{2} + 0 + 2\times 1\times\frac{1}{2} + 0 + 0$$
$$= 6 \; MPa$$

$$\sigma_{y'} = 4\times\frac{1}{2}+6\times\frac{1}{2}+0-2\times1\times\frac{1}{2}+0+0$$
$$= 4\ MPa$$
$$\sigma_{z'} = 0+0+8\times1+0+0+0$$
$$= 8\ MPa$$
$$\tau_{x'y'} = -4\times\frac{1}{2}+6\times\frac{1}{2}+0+1\left(\frac{1}{2}-\frac{1}{2}\right)+0+0$$
$$= 1\ MPa$$
$$\tau_{y'z'} = 0+0+0+0+0+2\left(-\frac{1}{\sqrt{2}}\right)$$
$$= -\sqrt{2}\ MPa$$
$$\tau_{x'z'} = 0+0+0+0+0+2\left(\frac{1}{\sqrt{2}}\right)$$
$$= \sqrt{2}\ MPa$$

Hence the new stress tensor becomes

$$\begin{bmatrix} 6 & 1 & \sqrt{2} \\ 1 & 4 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 8 \end{bmatrix} \quad MPa$$

Now, the new invariants are

$$I_1' = 6+4+8 = 18 \text{ MPa}$$
$$I_2' = 6\times4+4\times8+6\times8-1-2-2 = 99 \text{ MPa}$$
$$I_3' = 6\times30-1\times10+\sqrt{2}\left(-\frac{5}{\sqrt{2}}\right) = 160 \text{ MPa}$$

which remains unchanged. Hence proved.

Example 2.2

The state-of-stress at a point is given by the following array of terms

$$\begin{bmatrix} 9 & 6 & 3 \\ 6 & 5 & 2 \\ 3 & 2 & 4 \end{bmatrix} \boldsymbol{MPa}$$

Determine the principal stresses and principal directions.

Solution: The principal stresses are the roots of the cubic equation

$\sigma^3 - I_1\,\sigma^2 + I_2\,\sigma - I_3 = 0$

Here $I_1 = 9 + 5 + 4 = 18$ MPa

$$I_2 = 9\times5 + 5\times4 + 9\times4 - (6)^2 - (2)^2 - (3)^2 = 52\,\text{MPa}$$

$$I_3 = 9\times5\times4 - 9\times4 - 5\times9 - 4\times36 + 2\times6\times2\times3 = 27\,\text{MPa}$$

$\therefore$ The cubic equation becomes

$\sigma^3 - 18\sigma^2 + 52\sigma - 27 = 0$

The roots of the cubic equation are the principal stresses. Hence the three principal stresses are

$\sigma_1 = 14.554$ MPa; $\sigma_2 = 2.776$ MPa and $\sigma_3 = 0.669$ MPa

Now to find principal directions for major principal stress σ_1

$$\begin{vmatrix} (9-14.554) & 6 & 3 \\ 6 & (5-14.554) & 2 \\ 3 & 2 & (4-14.554) \end{vmatrix}$$

$$= \begin{vmatrix} -5.554 & 6 & 3 \\ 6 & -9.554 & 2 \\ 3 & 2 & -10.554 \end{vmatrix}$$

$$A = \begin{bmatrix} -9.554 & 2 \\ 2 & -10.554 \end{bmatrix} = 100.83 - 4 = 96.83$$

$$B = -\begin{bmatrix} 6 & 2 \\ 3 & -10.554 \end{bmatrix} = -(-63.324 - 6) = 69.324$$

$$C = \begin{bmatrix} 6 & -9.554 \\ 3 & 2 \end{bmatrix} = 12 + 28.662 = 40.662$$

$$\sqrt{A^2 + B^2 + C^2}$$

$$= \sqrt{(96.83)^2 + (69.324)^2 + (40.662)^2}$$

$= 125.83$

$$l_1 = \frac{A}{\sqrt{A^2 + B^2 + C^2}} = \frac{96.53}{125.83} = 0.769$$

$$m_1 = \frac{B}{\sqrt{A^2+B^2+C^2}} = \frac{69.324}{125.83} = 0.550$$

$$n_1 = \frac{C}{\sqrt{A^2+B^2+C^2}} = \frac{40.662}{125.84} = 0.325$$

Similarly, the principal stress directions for σ_2 stress and σ_3 stress are calculated.

Therefore, $l_2 = 0.596 \qquad l_3 = -0.226$

$m_2 = -0.800 \qquad m_3 = -0.177$

$n_2 = 0.057 \qquad n_3 = 0.944$

Example 2.3

At a point in the structural member, the stresses (in MPa) are represented as in Figure 2.18. Employ Mohr's circle to determine:

(a) the magnitude and orientation of the principal stresses

(b) the magnitude and orientation of the maximum shearing stresses and associated normal stresses.

In each case show the results on a properly oriented element.

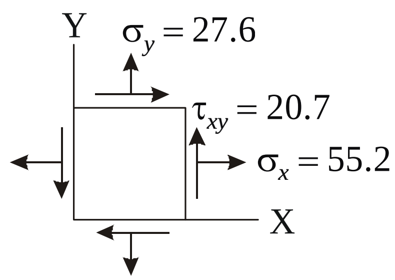


Figure 2.18

Solution: Centre of the Mohr's circle = OC

$$= \frac{27.6+55.2}{2} = 41.4\ MPa$$

(a) Principal stresses are represented by points A_1 and B_1. Hence the maximum and minimum principal stresses, referring to the circle are

$$\sigma_{1,2} = 41.4 \pm \sqrt{\frac{1}{4}(55.2-27.6)^2 + (20.7)^2}$$

$\sigma_1 = 66.3\ MPa$ and $\sigma_2 = 16.5\ MPa$

The planes on which the principal stresses act are given by

$$2\theta'_p = \tan^{-1}\frac{20.7}{13.8} = 56.30^0$$

and $2\theta''_p = 56.30 + 180 = 236.30^0$

Hence, $\theta'_p = 28.15^0$ and $\theta''_p = 118.15^0$

Mohr's circle clearly indicates that θ'_p locates the σ_1 plane.

(b) The maximum shearing stresses are given by points D and E. Thus

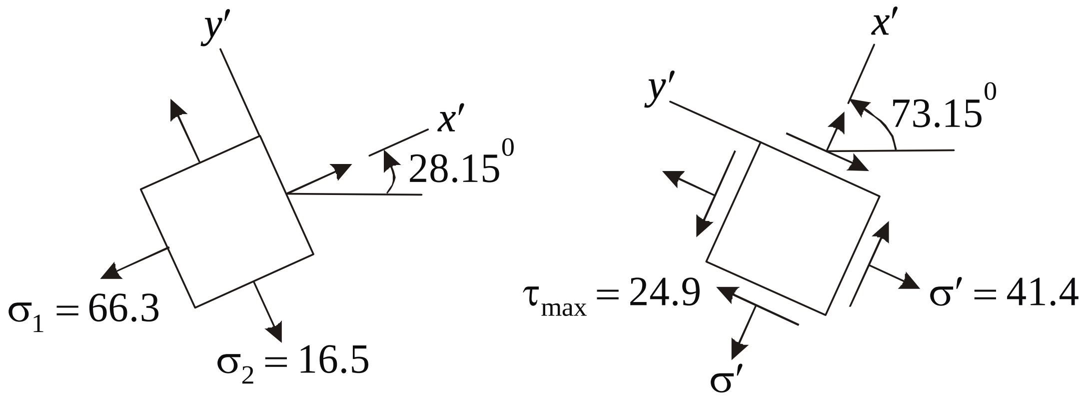
$$\tau_{max} = \pm\sqrt{\frac{1}{4}(55.2-27.6)^2 + (20.7)^2}$$

$$= \pm\ 24.9\ MPa$$

The planes on which these stresses act are represented by

$\theta'_s = 28.15^0 + 45^0 = 73.15^0$

and $\theta''_s = 163.15^0$



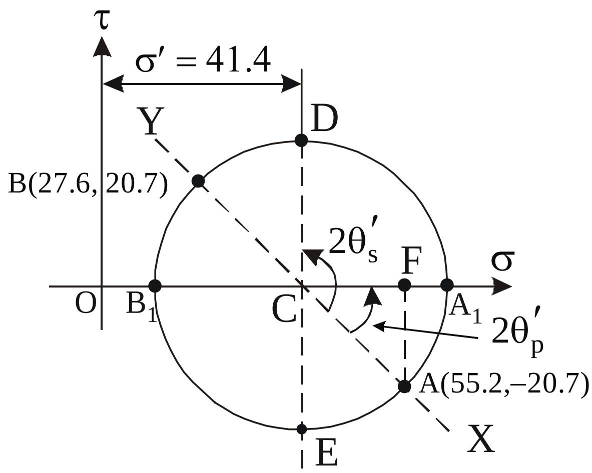


Figure 2.19 Mohr's stress circle

Example 2.4

The stress (in N/m²) acting on an element of a loaded body is shown in Figure 2.20. Apply Mohr's circle to determine the normal and shear stresses acting on a plane defined by $\theta = 30^0$.

Solution: The Mohr's circle drawn below describes the state of stress for the given element. Points A_1 and

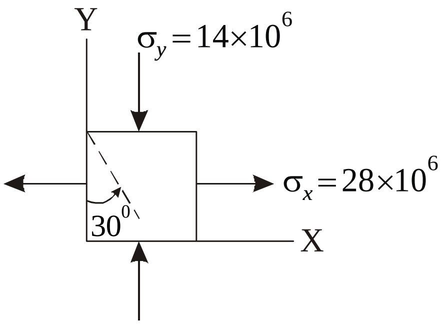


Figure 2.20

B_1 represent the stress components on the x and y faces, respectively.

The radius of the circle is $(14+28)\dfrac{10^6}{2} = 21\times10^6$. Corresponding to the 30^0 plane within the element, it is necessary to rotate through 60^0 counterclockwise on the circle to locate point A'. A 240^0 counterclockwise rotation locates point B'.

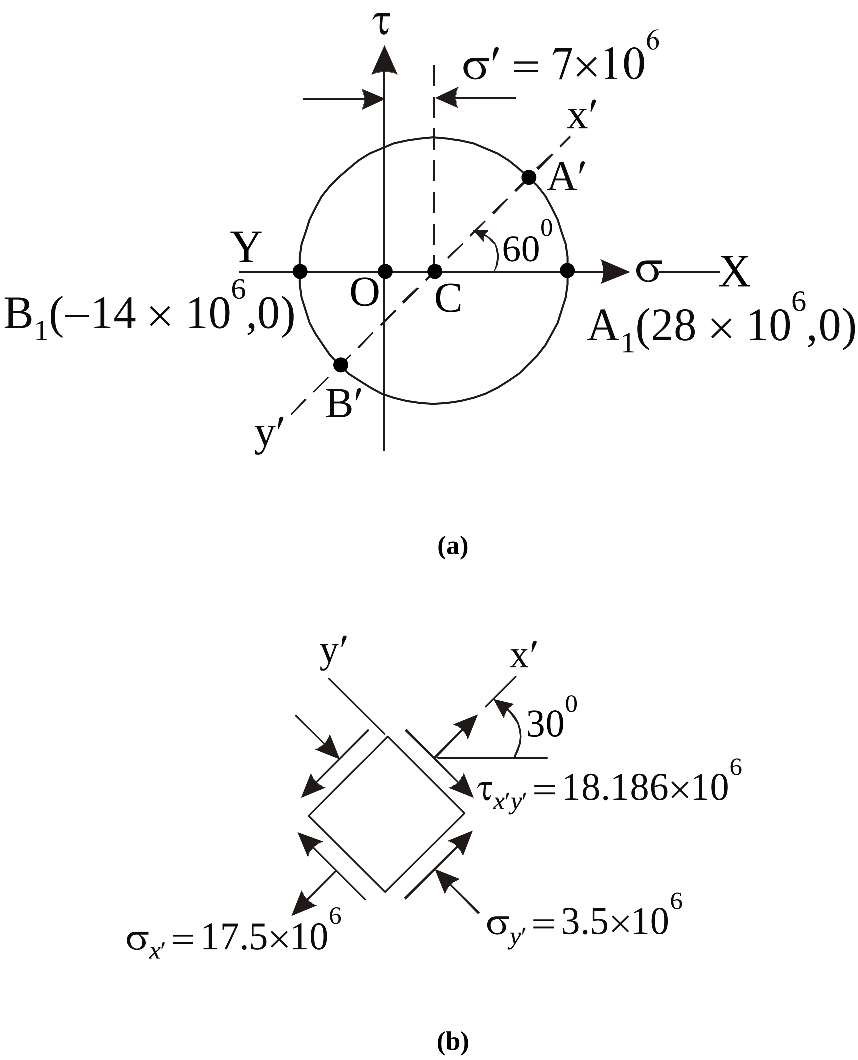


Figure 2.21 Mohr's stress circle

From the above Mohr's circle,

$$\sigma_{x'} = \left(7 + 21\cos 60^0\right)10^6 = 17.5\times 10^6\ N/m^2$$

$$\sigma_{y'} = -3.5\times 10^6\ N/m^2$$

$$\tau_{x'y'} = \pm 21\times 10^6 \sin 60^0 = \pm 18.86\times 10^6\ N/m^2$$

Example 2.5

A rectangular bar of metal of cross-section 30*mm*$\times$25*mm* is subjected to an axial tensile force of 180KN. Calculate the normal, shear and resultant stresses on a plane whose normal has the following direction cosines:

(i) $l = m = \frac{1}{\sqrt{2}}$ and $n = 0$

(ii) $l = m = n = \frac{1}{\sqrt{3}}$

Solution: Let normal stress acting on the cross-section is given by σ_y.

$$\therefore\ \sigma_y = \frac{\text{Axial load}}{\text{cross sectional area}}$$

$$= \frac{180\times 10^3}{30\times 25}$$

$$= 240\ N/mm^2$$

Now, By Cauchy's formula, the stress components along *x, y* and *z* co-ordinates are

$$T_x = \sigma_x l + \tau_{xy} m + \tau_{xz} n$$

$$T_y = \tau_{xy} l + \sigma_y m + \tau_{yz} n \qquad \text{(a)}$$

$$T_z = \tau_{xz} l + \tau_{yz} m + \sigma_z n$$

And the normal stress acting on the plane whose normal has the direction cosines *l, m* and *n* is,

$$\sigma = T_x l + T_y m + T_z n \qquad \text{(b)}$$

Case (i) For $l = m = \frac{1}{\sqrt{2}}\ \ and\ \ n = 0$

Here $\sigma_x = 0$, $\tau_{xy} = 0$, $\sigma_y = 240 N/mm^2$

$\tau_{xz} = 0$, $\tau_{yz} = 0$, $\sigma_z = 0$

Substituting the above in (a), we get

$$T_x = 0,\ T_y = \sigma_y m = \frac{240}{\sqrt{2}},\ T_z = 0$$

Substituting in (b), we get

$$\sigma = 0 + \frac{240}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right) + 0 = 120\, N/mm^2$$

Resultant Stress on the plane is

$$T = \sqrt{T_x^{\ 2} + T_y^{\ 2} + T_z^{\ 2}}$$

$$= \sqrt{0 + \left[\frac{240}{\sqrt{2}}\right]^2 + 0}$$

$$T = 169.706\, N/mm^2$$

But shear stress τ can be determined from the relation

$$T^2 = \sigma^2 + \tau^2$$

or $\tau = \sqrt{T^2 - \sigma^2}$

$$= \sqrt{(169.706)^2 - (120)^2}$$

$$\tau = 120\, N/mm^2$$

Case (ii) For $l = m = n = \frac{1}{\sqrt{3}}$

Again from (a),

$$T_x = 0,\ T_y = \sigma_y m = \frac{240}{\sqrt{3}},\ T_z = 0$$

Normal Stress = $\sigma = 0 + \frac{240}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right) + 0 = 80.00\, N/mm^2$

Resultant Stress on the plane is

$$T = \sqrt{T_x^{\ 2} + T_y^{\ 2} + T_z^{\ 2}}$$

$$T = \sqrt{0 + \left[\frac{240}{\sqrt{3}}\right]^2 + 0}$$

$$\tau = 113.13\ \ N/mm^2$$

Shear Stress = $\tau = \sqrt{(138.56)^2 - (80)^2}$

$$\tau = 113.13\, N/mm^2$$

Example 2.6

A body is subjected to three-dimensional forces and the state of stress at a point in it is represented as

$$\begin{bmatrix} 200 & 200 & 200 \\ 200 & -100 & 200 \\ 200 & 200 & -100 \end{bmatrix} MPa$$

Determine the normal stress, shearing stress and resultant stress on the octahedral plane.

Solution: For the octahedral plane, the direction cosines are

$$l = m = n = \frac{1}{\sqrt{3}}$$

Here $\sigma_x = 200\,MPa$

$\sigma_y = -100\,MPa$

$\sigma_y = -100\,MPa$

$\tau_{xy} = \tau_{yz} = \tau_{zx} = 200\,MPa$

Substituting the above in Cauchy's formula, we get

$$T_x = 200\left(\frac{1}{\sqrt{3}}\right) + 200\left(\frac{1}{\sqrt{3}}\right) + 200\left(\frac{1}{\sqrt{3}}\right) = 346.41\,MPa$$

$$T_y = 200\left(\frac{1}{\sqrt{3}}\right) - 100\left(\frac{1}{\sqrt{3}}\right) + 200\left(\frac{1}{\sqrt{3}}\right) = 173.20\,MPa$$

$$T_z = 200\left(\frac{1}{\sqrt{3}}\right) + 200\left(\frac{1}{\sqrt{3}}\right) - 100\left(\frac{1}{\sqrt{3}}\right) = 173.20\,MPa$$

Normal stress on the plane is given by

$$\sigma = T_x.l + T_y.m + T_z n$$

$$= 346.41\left(\frac{1}{\sqrt{3}}\right) + 173.20\left(\frac{1}{\sqrt{3}}\right) + 173.20\left(\frac{1}{\sqrt{3}}\right)$$

$$\sigma = 400\,MPa$$

Resultant Stress = $T = \sqrt{T_x^{\,2} + T_y^{\,2} + T_z^{\,2}}$

$$= \sqrt{(346.41)^2 + (173.20)^2 + (173.20)^2}$$

$$T = 424.26\,MPa$$

Also, Tangential Stress = $\tau = \sqrt{(424.26)^2 - (400)^2}$

$$= 141.41 \ MPa$$

Example 2.7
The state of stress at a point is given as follows:

$$\sigma_x = -800\,kPa, \quad \sigma_y = 1200kPa, \quad \sigma_z = -400kPa$$

$$\tau_{xy} = 400kPa, \quad \tau_{yz} = -600kPa, \quad \tau_{zx} = 500kPa$$

Determine (a) the stresses on a plane whose normal has direction cosines $l = \frac{1}{4},\ m = \frac{1}{2}$ **and (b) the normal and shearing stresses on that plane.**

Solution: We have the relation,

$$l^2 + m^2 + n^2 = 1$$

$$\therefore \left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + n^2 = 1$$

$$\therefore \ n = \frac{\sqrt{11}}{4}$$

(a) Using Cauchy's formula,

$$T_x = -800\left(\frac{1}{4}\right) + 400\left(\frac{1}{2}\right) + 500\left(\frac{\sqrt{11}}{4}\right) = 414.60\,kPa$$

$$T_y = 400\left(\frac{1}{4}\right) + 1200\left(\frac{1}{2}\right) - 600\left(\frac{\sqrt{11}}{4}\right) = 202.51\,kPa$$

$$T_z = 500\left(\frac{1}{4}\right) - 600\left(\frac{1}{2}\right) - 400\left(\frac{\sqrt{11}}{4}\right) = -506.66\,kPa$$

(b) Normal stress,

$$\sigma = T_x l + T_y m + T_z n$$

$$= 414.60\left(\frac{1}{4}\right) + 202.51\left(\frac{1}{2}\right) - 506.66\left(\frac{\sqrt{11}}{4}\right)$$

$$\sigma = -215.20\,kPa$$

Resultant Stress on the Plane = $T = \sqrt{(414.60)^2 + (202.51)^2 + (506.66)^2}$

$$= 685.28\ MPa$$

Shear Stress on the plane = $\tau = \sqrt{(685.28)^2 - (-215.20)^2}$
= 650.61 *kPa*

Example 2.8
Given the state of stress at a point as below

$$\begin{bmatrix} 100 & 80 & 0 \\ 90 & -60 & 0 \\ 0 & 0 & 40 \end{bmatrix} \textit{kPa}$$

Considering another set of coordinate axes, $x'\,y'\,z'$ in which z' coincides with *z* and x' is rotated by 30^0 anticlockwise from *x*-axis, determine the stress components in the new co-ordinates system.

Solution: The direction cosines for the transformation are given by

	X	*y*	*z*
x'	0.866	0.5	0
y'	-0.5	0.866	0
z'	0	0	1

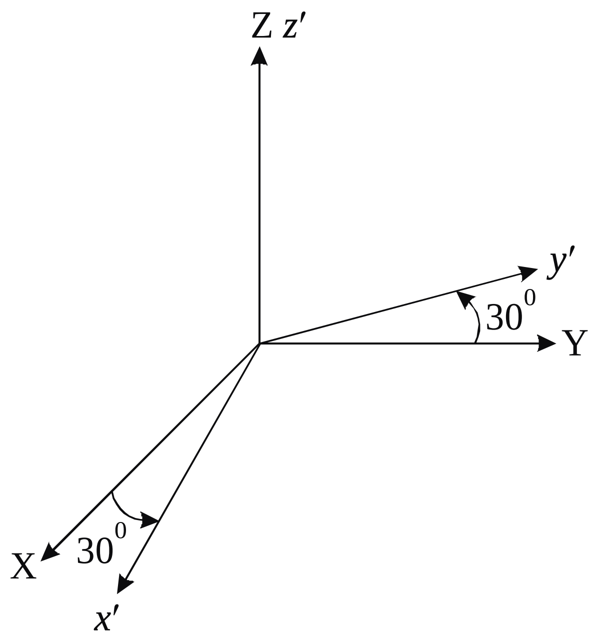


Figure 2.22 Co-ordinate system

Now using equations 2.21(a), 2.21(b), 2.21(c), 2.21(d), 2.21(e) and 2.21(f), we get

$\sigma_{x^1} = 100(0.866)^2 - 60(0.5)^2 + 0 + 2[80 \times 0.866 \times 0.5 + 0 + 0]$

$\sigma_{x'} = 129.3\,kPa$

$\sigma_{y'} = 100(-0.5)^2 - 60(0.866)^2 + 0 + 2[80(-0.5)(0.866) + 0 + 0]$

$\sigma_{y'} = -89.3\,kPa$

$\sigma_{z'} = 0 + 0 + 40(1)^2 + 2[0 + 0 + 0]$

$\sigma_{z'} = 40\,kPa$

$\tau_{x'y'} = 100(0.866)(-0.5) - 60(0.5)(0.866) + 0 + 80[(0.866 \times 0.866) + (-0.5)(0.5)] + 0 + 0$

$\tau_{x'y'} = -29.3\,kPa$

$\tau_{y'z'} = 0$ and $\tau_{z'x'} = 0$

Therefore the state of stress in new co-ordinate system is

$$\begin{bmatrix} 129.3 & -29.3 & 0 \\ -29.3 & -89.3 & 0 \\ 0 & 0 & 40 \end{bmatrix} (kPa)$$

Example 2.9

The stress tensor at a point is given by the following array

$$\begin{bmatrix} 50 & -20 & 40 \\ -20 & 20 & 10 \\ 40 & 10 & 30 \end{bmatrix} (kPa)$$

Determine the stress-vectors on the plane whose unit normal has direction cosines $\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}$

Solution: The stress vectors are given by

$$T_x = \sigma_x l + \tau_{xy} m + \tau_{xz} n \qquad \text{(a)}$$

$$T_y = \tau_{xy} l + \sigma_y m + \tau_{yz} n \qquad \text{(b)}$$

$$T_z = \tau_{xz} l + \tau_{yz} m + \sigma_z n \qquad \text{(c)}$$

Substituting the stress components in (a), (b) and (c) we get

$$T_x = 50\left(\frac{1}{\sqrt{2}}\right) - 20\left(\frac{1}{2}\right) + 40\left(\frac{1}{2}\right) = 45.35\,kPa$$

$$T_y = -20\left(\frac{1}{\sqrt{2}}\right) + 20\left(\frac{1}{2}\right) + 10\left(\frac{1}{2}\right) = 0.858\,kPa$$

$$T_z = 40\left(\frac{1}{\sqrt{2}}\right) + 10\left(\frac{1}{2}\right) + 30\left(\frac{1}{2}\right) = 48.28\,kPa$$

Now, Resultant Stress is given by

$$T = \left(45.35\hat{i} + 0.858\hat{j} + 48.28\hat{k}\right)kPa$$

Example 2.10

The Stress tensor at a point is given by the following array

$$\begin{bmatrix} 40 & 20 & 30 \\ 20 & 30 & 40 \\ 30 & 40 & 20 \end{bmatrix} (kPa)$$

Calculate the deviator and spherical stress tensors.

Solution: Mean Stress = $\sigma_m = \frac{1}{3}\left(\sigma_x + \sigma_y + \sigma_z\right)$

$$= \frac{1}{3}(40 + 30 + 20)$$

$$= 30\,kPa$$

Deviator stress tensor = $\begin{bmatrix} (\sigma_x - \sigma_m) & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma_m & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & (\sigma_z - \sigma_m) \end{bmatrix}$

$$= \begin{bmatrix} (40-30) & 20 & 30 \\ 20 & (30-30) & 40 \\ 30 & 40 & (20-30) \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 20 & 30 \\ 20 & 0 & 40 \\ 30 & 40 & -10 \end{bmatrix} kPa$$

Spherical Stress tensor = $\begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}$

$$= \begin{bmatrix} 30 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 30 \end{bmatrix} kPa$$

Example 2.11

The Stress components at a point in a body are given by

$$\sigma_x = 3xy^2z + 2x, \qquad \tau_{xy} = 0$$
$$\sigma_y = 5xyz + 3y \qquad \tau_{yz} = \tau_{xz} = 3xy^2z + 2xy$$
$$\sigma_z = x^2y + y^2z$$

Determine whether these components of stress satisfy the equilibrium equations or not as the point (1, -1, 2). If not then determine the suitable body force required at this point so that these stress components are under equilibrium.

Solution: The equations of equilibrium are given by

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \qquad \text{(a)}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0 \qquad \text{(b)}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \qquad \text{(c)}$$

Differentiating the stress components with respective axes, we get

$$\frac{\partial \sigma_x}{\partial x} = 3y^2z + 2, \quad \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \tau_{xz}}{\partial z} = 3xy^2$$

Substituting in (a), $3y^2z + 2 + 0 + 3xy^2$

At point (1, -1, 2), we get $3\times1\times2 + 2 + 3\times1\times1 = 11$ which is not equal to zero

Similarly,

$$\frac{\partial \sigma_y}{\partial y} = 5xz + 3, \quad \frac{\partial \tau_{yz}}{\partial z} = 3xy^2 + 0$$

$\therefore$ (ii) becomes $0 + 5xz + 3 + 3xy^2$

At point (1, -1, 2), we get $5\times1\times2 + 3 + 3\times1\times1 = 16$ which is not equal to zero

And $\frac{\partial \sigma_z}{\partial z} = y^2$, $\frac{\partial \tau_{yz}}{\partial y} = 6xyz + 2x$, $\frac{\partial \tau_{xz}}{\partial x} = 3y^2 z + 2y$

Therefore (iii) becomes $3y^2 z + 2y + 6xyz + 2x + y^2$

At the point (1, -1, 2), we get $3\times1\times2 + 2\times(-1) + 6\times1\times(-1)\times2 + 2\times1 + (-1)^2$ = -5 which is not equal to zero.

Hence the given stress components does not satisfy the equilibrium equations.

Recalling (a), (b) and (c) with body forces, the equations can be modified as below.

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x = 0 \qquad \text{(d)}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + F_y = 0 \qquad \text{(e)}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + F_z = 0 \qquad \text{(f)}$$

Where F_x, F_y *and* F_z are the body forces.

Substituting the values in (d), (e) and (f), we get body forces so that the stress components become under equilibrium.

Therefore,

$$3\times1\times2 + 2 + 3\times1\times1 + F_x = 0$$

$$\therefore \quad F_x = -11$$

Also, $5\times1\times2 + 3 + 3\times1\times1 + F_y = 0$

$$\therefore \quad F_y = -16$$

and $3\times1\times2 + 2\times(-1) + 6\times1\times(-1)\times2 + 2\times1 + (-1)^2 + F_z = 0$

$$\therefore F_z = 5$$

The body force vector is given by

$$\vec{F} = -11\hat{i} - 16\hat{j} + 5\hat{k}$$

Example 2.12

The rectangular stress components at a point in a three dimensional stress system are as follows.

$\sigma_x = 20N/mm^2$ $\sigma_y = -40N/mm^2$ $\sigma_z = 80N/mm^2$

$\tau_{xy} = 40N/mm^2$ $\tau_{yz} = -60N/mm^2$ $\tau_{zx} = 20N/mm^2$

Determine the principal stresses at the given point.

Solution: The principal stresses are the roots of the cubic equation

$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$

The three dimensional stresses can be expressed in the matrix form as below.

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} = \begin{bmatrix} 20 & 40 & 20 \\ 40 & -40 & -60 \\ 20 & -60 & 80 \end{bmatrix} N/mm^2$$

Here $I_1 = (\sigma_x + \sigma_y + \sigma_z)$

$= (20 - 40 + 80)$

$= 60$

$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau^2{}_{xy} - \tau^2{}_{yz} - \tau^2{}_{zx}$

$= (20(-40) + (-40)(80) + 80(20) - (40)^2 - (-60)^2 - (20)^2)$

$= -8000$

$I_3 = \sigma_x\sigma_y\sigma_z - \sigma_x\tau^2{}_{yz} - \sigma_y\tau^2{}_{zx} - \sigma_z\tau^2{}_{xy} + 2\tau_{xy}\tau_{yz}\tau_{xz}$

$= 20(-40)(80)-(20)(-60)^2-(-40)(20)^2-80(40)^2+2(40)(-60)(20)$

$= -344000$

Therefore Cubic equation becomes

$$\sigma^3 - 60\sigma^2 - 8000\sigma + 344000 = 0 \qquad \text{(a)}$$

Now $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

Or $\cos^3\theta - \frac{3}{4}\cos\theta - \frac{1}{4}\cos 3\theta = 0$ (b)

Put $\sigma = r\cos\theta + \frac{I_1}{3}$

i.e., $\sigma = r\cos\theta + \frac{60}{3}$

$\sigma = r\cos\theta + 20$

Substituting in (a), we get

$$(r\cos\theta + 20)^3 - 60(r\cos\theta + 20)^2 - 8000(r\cos\theta + 20) + 344000 = 0$$

$$(r\cos\theta + 20)^2 (r\cos\theta + 20) - 60(r\cos\theta + 20)^2 - 8000(r\cos\theta + 20) + 344000 = 0$$

$$(r^2\cos^2\theta + 400 + 40r\cos\theta)(r\cos\theta + 20) - 60(r^2\cos^2\theta + 400 + 40r\cos\theta)$$
$$-8000\,r\cos\theta - 160000 + 344000 = 0$$
$$r^3\cos^3\theta - 9200r\cos\theta + 168000 = 0$$

$$i.e., \cos^3\theta - \frac{1}{r^2}9200\cos\theta + \frac{168000}{r^3} = 0 \qquad \text{(c)}$$

Hence equations (b) and (c) are identical if

$$\frac{9200}{r^2} = \frac{3}{4}$$
$$\therefore r = \sqrt{\frac{9200 \times 4}{3}}$$
$$= 110.755$$

and $\frac{-\cos 3\theta}{4} = \frac{168000}{r^3}$

$$\therefore \quad -\cos 3\theta = \frac{168000 \times 4}{(110.755)^3} = 0.495$$

or $\cos 3\theta = -0.495$

$$\therefore 3\theta = 119.65 \quad or \quad \theta_1 = 39.9^0$$

$\theta_2 = 80.1^0$ and $\theta_3 = 159.9^0$

$$\therefore \quad \sigma = r_1\cos\theta_1 + \frac{I_1}{3}$$
$$= 110.755\cos(39.9) + \frac{60}{3}$$
$$= 104.96N/mm^2$$

$$\sigma_2 = r_2\cos\theta_2 + \frac{I_1}{3}$$
$$= 110.755\cos(80.1) + \frac{60}{3}$$
$$\sigma_2 = 39.04N/mm^2$$
$$\sigma_3 = r_3\cos\theta_3 + \frac{I_1}{3}$$

$$= 110.755\cos(159.9) + \frac{60}{3}$$

$$\sigma_3 = -84 N/mm^2$$

Example 2.13

At a point in a given material, the three dimensional state of stress is given by $\sigma_x = \sigma_y = \sigma_z = 10 N/mm^2, \tau_{xy} = 20 N/mm^2$ **and** $\tau_{yz} = \tau_{zx} = 10 N/mm^2$

Compute the principal planes if the corresponding principal stresses are

$\sigma_1 = 37.3 N/mm^2$, $\sigma_2 = -10 N/mm^2$, $\sigma_3 = 2.7 N/mm^2$

***Solution*:** The principal planes can be obtained by their direction Cosines *l, m and n* associated with each of the three principal stresses, σ_1, σ_2 *and* σ_3.

(a) To find Principal plane for Stress σ_1

$$\begin{vmatrix} (10-37.3) & 20 & 10 \\ 20 & (10-37.3) & 10 \\ 10 & 10 & (10-37.3) \end{vmatrix} = \begin{vmatrix} -27.3 & 20 & 10 \\ 20 & -27.3 & 10 \\ 10 & 10 & -27.3 \end{vmatrix}$$

Now, $A = \begin{vmatrix} -27.3 & 10 \\ 10 & -27.3 \end{vmatrix}$ = 745.29-100

A = 645.29

$$B = -\begin{vmatrix} 20 & 10 \\ 10 & -27.3 \end{vmatrix}$$

$$= -(-546 - 100)$$

B = 646

$$C = \begin{vmatrix} 20 & -27.3 \\ 10 & 10 \end{vmatrix}$$

= 200 + 270.3

C = 470.3

$$\sqrt{A^2 + B^2 + C^2} = \sqrt{(645.29)^2 + (646)^2 + (470.3)^2}$$

$$= 1027.08$$

$$\therefore l_1 = \frac{A}{\sqrt{A^2 + B^2 + C^2}} = \frac{645.29}{1027.08} = 0.628$$

$$m_1 = \frac{B}{\sqrt{A^2 + B^2 + C^2}} = \frac{646}{1027.08} = 0.628$$

$$n_1 = \frac{C}{\sqrt{A^2 + B^2 + C^2}} = \frac{470.3}{1027.08} = 0.458$$

(b) To find principal plane for Stress σ_2

$$\begin{vmatrix} (10+10) & 20 & 10 \\ 20 & (10+10) & 10 \\ 10 & 10 & (10+10) \end{vmatrix} = \begin{vmatrix} 20 & 20 & 10 \\ 20 & 20 & 10 \\ 10 & 10 & 20 \end{vmatrix}$$

$$A = \begin{vmatrix} 20 & 10 \\ 10 & 20 \end{vmatrix} = 400 - 100 = 300$$

$$B = -\begin{vmatrix} 20 & 10 \\ 10 & 20 \end{vmatrix} = -(400 - 100) = -300$$

$$C = \begin{vmatrix} 20 & 20 \\ 10 & 10 \end{vmatrix} = (200 - 200) = 0$$

$$\sqrt{A^2 + B^2 + C^2} = \sqrt{(300)^2 + (-300)^2 + (0)^2} = 424.26$$

$$\therefore l_2 = \frac{A}{\sqrt{A^2 + B^2 + C^2}} = \frac{300}{424.26} = 0.707$$

$$m_2 = \frac{B}{\sqrt{A^2 + B^2 + C^2}} = \frac{-300}{424.26} = -0.707$$

$$n_2 = \frac{C}{\sqrt{A^2 + B^2 + C^2}} = 0$$

(c) To find principal plane for Stress σ_3

$$\begin{vmatrix} (10-2.7) & 20 & 10 \\ 20 & (10-2.7) & 10 \\ 10 & 10 & (10-2.7) \end{vmatrix} = \begin{vmatrix} 7.3 & 20 & 10 \\ 20 & 7.3 & 10 \\ 10 & 10 & 7.3 \end{vmatrix}$$

$$A = \begin{vmatrix} 7.3 & 10 \\ 10 & 7.3 \end{vmatrix} = 53.29 - 100 = -46.71$$

$$B = -\begin{vmatrix} 20 & 10 \\ 10 & 7.3 \end{vmatrix} = -(146 - 100) = -46$$

$$C = \begin{vmatrix} 20 & 7.3 \\ 10 & 10 \end{vmatrix} = (200 - 73) = 127$$

$$\sqrt{A^2 + B^2 + C^2} = \sqrt{(-46.71)^2 + (46)^2 + (127)^2} = 142.92$$

$$\therefore l_3 = \frac{A}{\sqrt{A^2 + B^2 + C^2}} = \frac{-46.71}{142.92} = -0.326$$

$$m_3 = \frac{B}{\sqrt{A^2 + B^2 + C^2}} = \frac{-46}{142.92} = -0.322$$

$$n_3 = \frac{C}{\sqrt{A^2 + B^2 + C^2}} = \frac{127}{142.92} = 0.888$$